

Spectral analysis of the Neumann-Poincaré operator and characterization of the gradient blow-up*

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March 7, 2013

Abstract

When perfectly conducting or insulating inclusions are closely located, stress which is the gradient of the solution to the conductivity equation can be arbitrarily large as the distance between two inclusions tends to zero. It is important to precisely characterize the blow-up of the gradient. In this paper we show that the blow-up of the gradient can be characterized by a singular function defined by the single layer potential of an eigenfunction corresponding to the eigenvalue $1/2$ of a Neumann-Poincaré type operator defined on the boundaries of the inclusions. By comparing the singular function with the one corresponding to two disks osculating to the inclusions, we quantitatively characterize the blow-up of the gradient in terms of explicit functions.

Mathematics subject classification (MSC2000): 35J25, 73C40

Keywords: Neumann-Poincaré operator, gradient blow-up, perfectly conducting and insulating conductivity problems

1 Introduction

Let D_1 and D_2 be bounded simply connected domains in \mathbb{R}^d , $d = 2, 3$, whose boundary regularity will be specified later. Suppose that they are conductors, whose conductivity is k , $0 < k \neq 1 < \infty$, embedded in the background with conductivity 1. Let σ denote the conductivity distribution, *i.e.*,

$$\sigma = k\chi(D_1 \cup D_2) + \chi(\mathbb{R}^d \setminus (D_1 \cup D_2)), \quad (1.1)$$

*This work was supported by the ERC Advanced Grant Project MULTIMOD-267184 and NRF grants No. 2009-0085987, 2010-0004091, and 2010-0017532, and by Hankuk University of Foreign Studies Research Fund of 2012

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where χ is the characteristic function. We consider the following elliptic problem: for a given entire harmonic function h in \mathbb{R}^d ,

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0 & \text{in } \mathbb{R}^d, \\ u(\mathbf{x}) - h(\mathbf{x}) = O(|\mathbf{x}|^{1-d}) & \text{as } |\mathbf{x}| \rightarrow \infty. \end{cases} \quad (1.2)$$

Let

$$\epsilon := \text{dist}(D_1, D_2), \quad (1.3)$$

and assume that ϵ is small. We emphasize that the shapes of D_1 and D_2 do not depend on ϵ . More precisely, there are fixed domains \tilde{D}_1 and \tilde{D}_2 such that D_j is a translate of \tilde{D}_j , namely, there are vectors \mathbf{a}_1 and \mathbf{a}_2 such that

$$D_j = \tilde{D}_j + \mathbf{a}_j, \quad j = 1, 2. \quad (1.4)$$

The problem is to estimate $|\nabla u|$ in terms of ϵ when ϵ tends to 0, or to characterize the asymptotic singular behavior of ∇u as $\epsilon \rightarrow 0$.

If k stays away from 0 and ∞ , *i.e.*, $c_1 < k < c_2$ for some positive constants c_1 and c_2 , then $|\nabla u|$ is bounded regardless of ϵ as was proved in [10, 19, 18]. In fact, it is proved that the $\mathcal{C}^{1,\alpha}$ norm of u is bounded regardless of ϵ when ∂D_1 and ∂D_2 are $\mathcal{C}^{2,\alpha}$ smooth. However, if k degenerates to either ∞ (perfectly conducting case) or 0 (insulating case), the ellipticity holds only outside D_1 and D_2 and completely different phenomena occur.

When $k = \infty$, the problem becomes

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^d \setminus \overline{(D_1 \cup D_2)}, \\ u = \lambda_i \text{ (constant)} & \text{on } \partial D_i, \quad i = 1, 2, \\ u(\mathbf{x}) - h(\mathbf{x}) = O(|\mathbf{x}|^{1-d}) & \text{as } |\mathbf{x}| \rightarrow \infty, \end{cases} \quad (1.5)$$

where the constants λ_i are determined by the conditions

$$\int_{\partial D_1} \frac{\partial u}{\partial \nu^{(1)}} \Big|_+ = \int_{\partial D_2} \frac{\partial u}{\partial \nu^{(2)}} \Big|_+ = 0, \quad (1.6)$$

with $\nu^{(j)}$ being the outward unit normal to ∂D_j , $j = 1, 2$. In that case, ∇u may blow up as ϵ tends to 0.

As shown in [17, 11, 5, 3, 23, 24, 7], in two dimensions the generic rate of gradient blow-up is $\epsilon^{-1/2}$, while it is $|\epsilon \log \epsilon|^{-1}$ in three dimensions [7, 8, 20]. The blow-up of the gradient may or may not occur depending on the background potential (the harmonic function h in (1.2)) and those background potentials which actually make the gradient blow up are characterized in [4] when D_1 and D_2 are disks. In two dimensions, the perfectly insulating case, where $k = 0$, can be dealt with using the conjugate relation (see [17, 5]) and in this case the blow-up rate is also $\epsilon^{-1/2}$. It is a challenging open problem to clarify whether $|\nabla u|$ may blow up or not in the insulating case in three dimensions and to find the blow-up rate if the blow-up occurs. It is also a quite interesting problem to clarify the dependence of $|\nabla u|$ on k as $k \rightarrow \infty$ or $k \rightarrow 0$. In this relation we mention that a precise dependence on k when D_1 and D_2 are disks was shown in [5, 3]. It is worth mentioning that a similar blow-up phenomenon for the p -Laplacian equation was investigated in [12].

Above mentioned results for $k = \infty$ are estimates of $|\nabla u|$ in upper and lower bounds sense, namely,

$$C_1\psi(\epsilon) \leq \|\nabla u\|_\infty \leq C_2\psi(\epsilon) \quad (1.7)$$

for some implicit constants C_1 and C_2 where $\psi(\epsilon) = \epsilon^{-1/2}$ in two dimensions and $\psi(\epsilon) = |\epsilon \log \epsilon|^{-1}$ in three dimensions. The constant C_1 can be zero or positive depending on the background potential h . In order to have a better understanding of the concentration of the gradient it is desirable to pursue deeper investigation on the blow-up nature of ∇u . In this direction there is a recent work [13] where the blow-up nature of ∇u is characterized in terms of an explicit singular function. It is shown that if $D_1 = B_1$ and $D_2 = B_2$ are circular inclusions of radius r_1 and r_2 , respectively, and $k = \infty$, then

$$u(\mathbf{x}) = \frac{2r_1r_2}{r_1 + r_2}(\mathbf{n} \cdot \nabla h)(\mathbf{p}) (\ln |\mathbf{x} - \mathbf{p}_1| - \ln |\mathbf{x} - \mathbf{p}_2|) + r(\mathbf{x}), \quad (1.8)$$

for $\mathbf{x} \in \mathbb{R}^2 \setminus (B_1 \cup B_2)$, where $\mathbf{p}_1 \in D_1$ is the fixed point of R_1R_2 where R_j is the reflection with respect to ∂B_j , $j = 1, 2$, $\mathbf{p}_2 \in B_2$ is the fixed point of R_2R_1 , \mathbf{n} is the unit vector in the direction of $\mathbf{p}_2 - \mathbf{p}_1$, and \mathbf{p} is the middle point of the shortest line segment connecting ∂B_1 and ∂B_2 . In (1.8), ∇r is bounded independently of ϵ and thus the blow-up of ∇u is completely characterized by the singular function

$$q_B(\mathbf{x}) := \frac{1}{2\pi} (\ln |\mathbf{x} - \mathbf{p}_1| - \ln |\mathbf{x} - \mathbf{p}_2|). \quad (1.9)$$

The purpose of this paper is to establish a characterization of the blow-up of ∇u similar to (1.8) when D_1 and D_2 are strictly convex simply connected domains in \mathbb{R}^2 . In doing so, the Neumann-Poincaré (NP) operator denoted by \mathbb{K}^* and defined on $L^2(\partial D_1) \times L^2(\partial D_2)$ plays a crucial role. The NP operator is a classical notion and appears naturally when we solve the boundary value problems using layer potentials. It also appears naturally when we solve the transmission problem (1.2). See the next section for a definition of the NP operator for the problem of this paper. This operator has $1/2$ as an eigenvalue with multiplicity 2. (If the inclusion has N simply connected components, then the multiplicity of $1/2$ is N .) If two inclusions are disks, then $(\frac{\partial q_B}{\partial \nu^{(1)}}|_{\partial D_1}, \frac{\partial q_B}{\partial \nu^{(2)}}|_{\partial D_2})^T$, where q_B is given by (1.9), is an eigenfunction of the NP operator on $L^2(\partial D_1) \times L^2(\partial D_2)$ corresponding to $1/2$. (This fact was also observed in [9].) Here, T denotes the transpose.

Let $\mathbf{g} = (g^{(1)}, g^{(2)})^T$ be the eigenfunction of \mathbb{K}^* on $L^2(\partial D_1) \times L^2(\partial D_2)$ corresponding to the eigenvalue $1/2$ and satisfying

$$\int_{\partial D_1} g^{(1)} d\sigma = 1, \quad \int_{\partial D_2} g^{(2)} d\sigma = -1. \quad (1.10)$$

We will prove that such an eigenfunction does exist. Let h be the background harmonic function introduced in (1.2) and let

$$\mathbf{h} := \begin{bmatrix} h|_{\partial D_1} \\ h|_{\partial D_2} \end{bmatrix}. \quad (1.11)$$

Let $\langle \mathbf{h}, \mathbf{g} \rangle$ be the inner product on $L^2(\partial D_1) \times L^2(\partial D_2)$, i.e.,

$$\langle \mathbf{h}, \mathbf{g} \rangle = \int_{\partial D_1} h g^{(1)} d\sigma + \int_{\partial D_2} h g^{(2)} d\sigma. \quad (1.12)$$

With these notions in hand, we can state the main result of this paper.

Theorem 1.1 *Let D_1 and D_2 be strictly convex simply connected domains in \mathbb{R}^2 with $\mathcal{C}^{2,\alpha}$ smooth boundaries for some $\alpha \in (0, 1]$. Let $\mathbf{z}_1 \in \partial D_1$ and $\mathbf{z}_2 \in \partial D_2$ be the closest points, and let $\epsilon := \text{dist}(D_1, D_2) = |\mathbf{z}_1 - \mathbf{z}_2|$, κ_j be the curvature of ∂D_j at \mathbf{z}_j , B_j be the disk osculating to D_j at \mathbf{z}_j , $j = 1, 2$, and q_B be the singular function in (1.9) associated with disks B_1 and B_2 . Then, the solution u to (1.5) satisfies*

$$u(\mathbf{x}) = -\frac{\sqrt{2\pi}\langle \mathbf{h}, \mathbf{g} \rangle}{\sqrt{\epsilon(\kappa_1 + \kappa_2)}} \alpha_\epsilon q_B(\mathbf{x}) + r(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2 \setminus (D_1 \cup D_2), \quad (1.13)$$

where α_ϵ is a constant bounded independently of ϵ of the form

$$\alpha_\epsilon = \begin{cases} 1 + O(\epsilon^{\alpha/2}) & \text{if } \alpha \in (0, 1), \\ 1 + O(|\sqrt{\epsilon} \ln \epsilon|) & \text{if } \alpha = 1, \end{cases} \quad \text{as } \epsilon \rightarrow 0, \quad (1.14)$$

and

$$\|\nabla r\|_{L^\infty(\Omega)} \leq C \quad (1.15)$$

for some constant C independent of ϵ . Here $\Omega = \Omega_1 \setminus (D_1 \cup D_2)$ and Ω_1 is an open set containing $\overline{D_1 \cup D_2}$.

We emphasize that (1.13) is a pointwise relation and hence describes the behavior of $\nabla u(\mathbf{x})$ in terms of the gradient of the function $q_B(\mathbf{x})$. One can see from the explicit expression (1.9) that $|\nabla q_B|$ attains its maximum at \mathbf{z}_1 and \mathbf{z}_2 , and that

$$|\nabla q_B(\mathbf{z}_j)| = \frac{\sqrt{\kappa_1 + \kappa_2}}{\sqrt{2\pi}} \frac{1}{\sqrt{\epsilon}} + O(1). \quad (1.16)$$

(See (5.5).) So, (1.13) shows that $|\nabla u|$ is bounded regardless of ϵ if $\langle \mathbf{h}, \mathbf{g} \rangle = 0$. Moreover, it yields a new improved estimate:

$$\|\nabla u\|_{L^\infty(\mathbb{R}^2 \setminus (D_1 \cup D_2))} = \frac{\alpha_\epsilon |\langle \mathbf{h}, \mathbf{g} \rangle|}{\epsilon} + O(1), \quad \text{as } \epsilon \rightarrow 0. \quad (1.17)$$

Since

$$|\langle \mathbf{h}, \mathbf{g} \rangle| \leq C\sqrt{\epsilon} \quad (1.18)$$

for some constant C independent of ϵ as shown in (5.24), we can also infer from (1.17) that the generic rate of blow-up is $\epsilon^{-1/2}$.

The (global) strict convexity assumption of D_1 and D_2 in Theorem 1.1 can be relaxed a little. Instead, if we let $\mathbf{z} = \frac{\mathbf{z}_1 + \mathbf{z}_2}{2}$, then it is enough to assume that there are δ and c_0 such that $\partial D_j \cap B_\delta(\mathbf{z})$ is strictly convex for $j = 1, 2$, and

$$\text{dist}(\partial D_1 \setminus B_\delta(\mathbf{z}), \partial D_2 \setminus B_\delta(\mathbf{z})) \geq c_0. \quad (1.19)$$

This can be shown by exactly the same proofs as in this paper.

We also obtain similar results for the insulating case and boundary value problems. The problem for the insulating case, obtained by taking the limit as $k \rightarrow 0$ of (1.2), is given by

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^d \setminus \overline{(D_1 \cup D_2)}, \\ \frac{\partial u}{\partial \nu^{(i)}} \Big|_+ = 0 & \text{on } \partial D_i, \ i = 1, 2, \\ u(\mathbf{x}) - h(\mathbf{x}) = O(|\mathbf{x}|^{1-d}) & \text{as } |\mathbf{x}| \rightarrow \infty. \end{cases} \quad (1.20)$$

If u is the solution to (1.20), then its harmonic conjugate u^\perp is the solution to (1.5) with h replaced with its harmonic conjugate h^\perp . Therefore, we may apply Theorem 1.1 to u^\perp to obtain an analogous result for u .

On the other hand, if we consider the boundary value problem

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in } \Omega, \quad (1.21)$$

with either Dirichlet or Neumann boundary conditions on the boundary $\partial\Omega$ of a smooth domain Ω containing D_1 and D_2 and $k \rightarrow 0$ or ∞ , then, in this case, the harmonic function h defined by

$$h(\mathbf{x}) = -\frac{1}{2\pi} \int_{\partial\Omega} \ln |\mathbf{x} - \mathbf{y}| \frac{\partial u}{\partial \nu}(\mathbf{y}) d\sigma(\mathbf{y}) - \frac{1}{2\pi} \int_{\partial\Omega} \frac{\langle \mathbf{x} - \mathbf{y}, \nu(\mathbf{y}) \rangle}{|\mathbf{x} - \mathbf{y}|^2} u(\mathbf{y}) d\sigma(\mathbf{y}), \quad (1.22)$$

with ν being the outward normal to $\partial\Omega$, plays the role of h in the whole space problem (1.5) or (1.20). A similar result on the characterization of the gradient blow-up can be obtained using exactly the same arguments as for the whole space problem.

The results of this paper can be applied for solving two longstanding problems. The first one is the study of material failure. In fact, the problem of estimation of the gradient blow-up was raised by Babuška in relation to the study of material failure of composites [6]. In composites which consist of inclusions and the matrix, some inclusions may be closely located and stress occurs in between them. The problems (1.2), (1.5) and (1.20) are conductivity or anti-plane elasticity equation, and ∇u represents the shear stress tensor. So results of this paper provide clear quantitative understanding of the stress concentration, which will be a fundamental ingredient in the study of material failure.

The second application is computation of the electrical field in the presence of closely located inclusions with extreme conductivities (0 or ∞) which is known to be a hard problem. Because $|\nabla u|$ becomes arbitrarily large, we need fine meshes to compute ∇u numerically. Since (1.13) for example provides complete characterization of the singular behavior of ∇u , the complexity of computation can be greatly reduced by removing the singular term there. In fact, effectiveness of this scheme is already demonstrated in [13] when inclusions are disks using (1.8). It is worth mentioning that unlike (1.8) where the coefficient of q_B is explicitly determined by h , computation of the constant $\alpha_\epsilon(\mathbf{h}, \mathbf{g})$ in (1.13) may cause a problem when inclusions are of general shape. We will investigate this issue in a forthcoming work.

This paper is organized as follows. In the next section we introduce the single layer potential and define the Neumann-Poincaré operator. In section 3, we construct eigenfunctions of the NP operator corresponding to the eigenvalue $1/2$ and prove that its multiplicity is 2. In section 4, we construct a singular function using eigenfunctions constructed in the previous section and characterize the gradient blow-up in terms of the singular function. In section 5, we estimate the potential difference of the solution to (1.5). Section 6 is to prove Theorem 1.1. Sections 7 and 8 are for the insulating case and the boundary value problem, respectively. In the last section we prove a lemma used in Section 6.

2 Preliminaries

Let D be a bounded simply connected domain in \mathbb{R}^d , $d = 2, 3$, with a Lipschitz boundary. The single layer potential $\mathcal{S}_D[\varphi]$ of a function $\varphi \in L^2(\partial D)$ is defined as

$$\mathcal{S}_D[\varphi](\mathbf{x}) = \int_{\partial D} \Gamma(\mathbf{x} - \mathbf{y}) \varphi(\mathbf{y}) \, d\sigma(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^d,$$

where Γ is the fundamental solution to the Laplacian, *i.e.*,

$$\Gamma(\mathbf{x}) = \begin{cases} \frac{1}{2\pi} \ln |\mathbf{x}|, & d = 2, \\ -\frac{1}{4\pi} |\mathbf{x}|^{-1}, & d = 3. \end{cases} \quad (2.1)$$

Then, for $\varphi \in L^2(\partial D)$, we have

$$\left. \frac{\partial}{\partial \nu} \mathcal{S}_D[\varphi] \right|_{\pm}(\mathbf{x}) = \left(\pm \frac{1}{2} + \mathcal{K}_D^* \right) [\varphi](\mathbf{x}) \quad \text{a.e. } \mathbf{x} \in \partial D, \quad (2.2)$$

where

$$\mathcal{K}_D^*[\varphi](\mathbf{x}) = \int_{\partial D} \frac{\partial \Gamma(\mathbf{x} - \mathbf{y})}{\partial \nu(\mathbf{x})} \varphi(\mathbf{y}) \, d\sigma(\mathbf{y}), \quad \mathbf{x} \in \partial D. \quad (2.3)$$

Here, $\frac{\partial}{\partial \nu}$ denotes the normal derivative and the subscripts $+$ and $-$ represent the limits from outside and inside D , respectively. The operator \mathcal{K}_D^* is called the Neumann-Poincaré (NP) operator.

We now consider the configuration where there are two inclusions D_1 and D_2 which are closely located. Suppose that the conductivity of the inclusions is $k \neq 1$ while that of the background is 1, so that the conductivity distribution is given by (1.1). For a given entire harmonic function h in \mathbb{R}^d , we consider the problem (1.2).

It is known (see for example [14, 15]) that the solution u to (1.5) can be represented as

$$u(\mathbf{x}) = h(\mathbf{x}) + \mathcal{S}_{D_1}[\varphi^{(1)}](\mathbf{x}) + \mathcal{S}_{D_2}[\varphi^{(2)}](\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d \setminus (D_1 \cup D_2) \quad (2.4)$$

for a pair of functions $(\varphi^{(1)}, \varphi^{(2)}) \in L_0^2(\partial D_1) \times L_0^2(\partial D_2)$ (L_0^2 denotes the set of L^2 functions with mean zero). Since u is constant on ∂D_1 and ∂D_2 , we have

$$\left. \frac{\partial}{\partial \nu^{(j)}} (\mathcal{S}_{D_1}[\varphi^{(1)}] + \mathcal{S}_{D_2}[\varphi^{(2)}]) \right|_- = -\frac{\partial h}{\partial \nu^{(j)}} \quad \text{on } \partial D_j, \quad j = 1, 2,$$

which, according to (2.2), may be written as

$$\left(\frac{1}{2} I - \mathcal{K}_{D_1}^* \right) [\varphi^{(1)}] - \frac{\partial}{\partial \nu^{(1)}} \mathcal{S}_{D_2}[\varphi^{(2)}] = \frac{\partial h}{\partial \nu^{(1)}} \quad \text{on } \partial D_1, \quad (2.5)$$

$$-\frac{\partial}{\partial \nu^{(2)}} \mathcal{S}_{D_1}[\varphi^{(1)}] + \left(\frac{1}{2} I - \mathcal{K}_{D_2}^* \right) [\varphi^{(2)}] = \frac{\partial h}{\partial \nu^{(2)}} \quad \text{on } \partial D_2. \quad (2.6)$$

Here, $\frac{\partial h}{\partial \nu^{(j)}}$ denotes the outward normal derivative on ∂D_j , $j = 1, 2$. This system of integral equations can be written in a condensed form as

$$\left(\frac{1}{2} \mathbb{I} - \mathbb{K}^* \right) [\varphi] = \partial h, \quad (2.7)$$

where

$$\mathbb{I} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad \mathbb{K}^* := \begin{bmatrix} \mathcal{K}_{D_1}^* & \frac{\partial}{\partial \nu^{(1)}} \mathcal{S}_{D_2} \\ \frac{\partial}{\partial \nu^{(2)}} \mathcal{S}_{D_1} & \mathcal{K}_{D_2}^* \end{bmatrix} \quad (2.8)$$

(with I being the identity operator), and

$$\varphi := \begin{bmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{bmatrix}, \quad \partial h := \begin{bmatrix} \frac{\partial h}{\partial \nu^{(1)}} \\ \frac{\partial h}{\partial \nu^{(2)}} \end{bmatrix}.$$

If there are N simply connected inclusions, say D_1, \dots, D_N , then the corresponding NP operator \mathbb{K}^* is defined by

$$\mathbb{K}^* := \begin{bmatrix} \mathcal{K}_{D_1}^* & \frac{\partial}{\partial \nu^{(1)}} \mathcal{S}_{D_2} & \cdots & \frac{\partial}{\partial \nu^{(1)}} \mathcal{S}_{D_N} \\ \frac{\partial}{\partial \nu^{(2)}} \mathcal{S}_{D_1} & \mathcal{K}_{D_2}^* & \cdots & \frac{\partial}{\partial \nu^{(2)}} \mathcal{S}_{D_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \nu^{(N)}} \mathcal{S}_{D_1} & \frac{\partial}{\partial \nu^{(N)}} \mathcal{S}_{D_2} & \cdots & \mathcal{K}_{D_N}^* \end{bmatrix}. \quad (2.9)$$

We make note of some important properties of the NP operator \mathbb{K}^* whose proofs can be found in [1]. Let $\mathcal{H} := L^2(\partial D_1) \times L^2(\partial D_2)$ and $\mathcal{H}_0 := L_0^2(\partial D_1) \times L_0^2(\partial D_2)$. We have

- \mathbb{K}^* maps \mathcal{H} into itself, and \mathcal{H}_0 into itself.
- For any λ with $\lambda \leq -1/2$ or $\lambda > 1/2$, $\lambda \mathbb{I} - \mathbb{K}^*$ is invertible on \mathcal{H} .
- $\frac{1}{2} \mathbb{I} - \mathbb{K}^*$ is invertible on \mathcal{H}_0 .
- All the eigenvalues of \mathbb{K}^* belong to $(-1/2, 1/2]$.

One fact of crucial importance is that \mathbb{K}^* can be symmetrized. To see this we introduce the operator \mathbb{S} acting on \mathcal{H} as

$$\mathbb{S} = \begin{bmatrix} \mathcal{S}_{D_1} & \mathcal{S}_{D_2} \\ \mathcal{S}_{D_1} & \mathcal{S}_{D_2} \end{bmatrix}. \quad (2.10)$$

It is worth making a remark on the operators off diagonal. For example, \mathcal{S}_{D_2} on the top right corner is an operator from $L^2(\partial D_2)$ into $L^2(\partial D_1)$. It is proved in [1] based on a result in [16] that $-\mathbb{S}$ is positive semi-definite and self-adjoint, $\mathbb{S}\mathbb{K}^*$ is self-adjoint, and hence there is a self-adjoint operator \mathbb{A} on \mathcal{H} such that

$$\sqrt{-\mathbb{S}}\mathbb{K}^* = \mathbb{A}\sqrt{-\mathbb{S}}. \quad (2.11)$$

In other words, \mathbb{K}^* is self-adjoint with respect to the inner product

$$\langle \varphi, \psi \rangle_{\mathbb{A}} := -\langle \mathbb{S}[\varphi], \psi \rangle, \quad \varphi, \psi \in \mathcal{H}. \quad (2.12)$$

3 Eigenfunctions of \mathbb{K}^*

We now construct eigenfunctions of \mathbb{K}^* corresponding to $1/2$. Our construction plays an essential role in understanding the blow-up of the gradient. We first prove the following lemma.

Lemma 3.1 *For $i = 1, 2$, there is a unique solution v_i to*

$$\begin{cases} \Delta v_i = 0 & \text{in } \mathbb{R}^d \setminus \overline{D_1 \cup D_2}, \\ v_i = \lambda_j \text{ (constant)} & \text{on } \partial D_j, \ j = 1, 2, \\ \int_{\partial D_i} \frac{\partial v_i}{\partial \nu^{(i)}} \Big|_+ d\sigma \neq 0, \int_{\partial D_j} \frac{\partial v_i}{\partial \nu^{(j)}} \Big|_+ d\sigma = 0 & \text{if } j \neq i, \\ v_i(\mathbf{x}) - \mathcal{S}_{D_i}[1](\mathbf{x}) = O(|\mathbf{x}|^{1-d}) & \text{as } |\mathbf{x}| \rightarrow \infty. \end{cases} \quad (3.1)$$

Proof. We first observe that

$$\int_{\partial D_j} \frac{\partial}{\partial \nu^{(j)}} \mathcal{S}_{D_i}[1] \Big|_- d\sigma = 0, \quad j = 1, 2.$$

Since $\frac{1}{2}\mathbb{I} - \mathbb{K}^*$ is invertible on \mathcal{H}_0 , there exists a unique solution $(\psi_i^{(1)}, \psi_i^{(2)})^T \in \mathcal{H}_0$ such that

$$\left(\frac{1}{2}\mathbb{I} - \mathbb{K}^* \right) \begin{bmatrix} \psi_i^{(1)} \\ \psi_i^{(2)} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial \nu^{(1)}} \mathcal{S}_{D_i}[1] \Big|_- \\ \frac{\partial}{\partial \nu^{(2)}} \mathcal{S}_{D_i}[1] \Big|_- \end{bmatrix} \in \mathcal{H}_0. \quad (3.2)$$

Let δ_{ij} be the Kronecker symbol and define

$$\begin{bmatrix} \varphi_i^{(1)} \\ \varphi_i^{(2)} \end{bmatrix} := \begin{bmatrix} \psi_i^{(1)} \\ \psi_i^{(2)} \end{bmatrix} + \begin{bmatrix} \delta_{i1} \\ \delta_{i2} \end{bmatrix}. \quad (3.3)$$

Since

$$\begin{bmatrix} \frac{\partial}{\partial \nu^{(1)}} \mathcal{S}_{D_i}[1] \Big|_- \\ \frac{\partial}{\partial \nu^{(2)}} \mathcal{S}_{D_i}[1] \Big|_- \end{bmatrix} = \left(-\frac{1}{2}\mathbb{I} + \mathbb{K}^* \right) \begin{bmatrix} \delta_{i1} \\ \delta_{i2} \end{bmatrix},$$

we have

$$\left(\frac{1}{2}\mathbb{I} - \mathbb{K}^* \right) \begin{bmatrix} \varphi_i^{(1)} \\ \varphi_i^{(2)} \end{bmatrix} = 0. \quad (3.4)$$

Define

$$v_i(\mathbf{x}) := \mathcal{S}_{D_1}[\varphi_i^{(1)}](\mathbf{x}) + \mathcal{S}_{D_2}[\varphi_i^{(2)}](\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d \setminus (D_1 \cup D_2). \quad (3.5)$$

We now show that v_i is the desired function. Because of (3.4), we have

$$\frac{\partial}{\partial \nu^{(j)}} (\mathcal{S}_{D_1}[\varphi_i^{(1)}] + \mathcal{S}_{D_2}[\varphi_i^{(2)}]) \Big|_- = 0 \quad \text{on } \partial D_j, \quad j = 1, 2,$$

and hence, $\mathcal{S}_{D_1}[\varphi_i^{(1)}] + \mathcal{S}_{D_2}[\varphi_i^{(2)}]$ is constant in D_1 and D_2 . Thus, v_i is constant on ∂D_1 and ∂D_2 .

Since $(\psi_i^{(1)}, \psi_i^{(2)})^T \in \mathcal{H}_0$, we have

$$\int_{\partial D_j} \frac{\partial}{\partial \nu^{(j)}} (\mathcal{S}_{D_1}[\psi_i^{(1)}] + \mathcal{S}_{D_2}[\psi_i^{(2)}]) \Big|_+ d\sigma = 0, \quad j = 1, 2.$$

On the other hand, we have

$$\int_{\partial D_i} \frac{\partial}{\partial \nu^{(i)}} \mathcal{S}_{D_i}[1] \Big|_+ d\sigma = \int_{\partial D_i} 1 + \frac{\partial}{\partial \nu^{(i)}} \mathcal{S}_{D_i}[1] \Big|_- d\sigma = |\partial D_i|,$$

and

$$\int_{\partial D_j} \frac{\partial}{\partial \nu^{(j)}} \mathcal{S}_{D_i}[1] \Big|_+ d\sigma = \int_{\partial D_j} \frac{\partial}{\partial \nu^{(j)}} \mathcal{S}_{D_i}[1] \Big|_- d\sigma = 0$$

if $j \neq i$. Here $|\partial D_j|$ denotes the area (or length) of ∂D_j . Thus we have the third line in (3.1). This completes the proof. \square

As an immediate consequence we have the following theorem.

Theorem 3.2 *The eigenvalue $\frac{1}{2}$ of \mathbb{K}^* has multiplicity 2.*

Proof. The identity (3.4) shows that $\varphi_j := (\varphi_j^{(1)}, \varphi_j^{(2)})^T$, $j = 1, 2$, are two eigenfunctions of \mathbb{K}^* corresponding to $1/2$. We have from (3.3) that

$$\int_{\partial D_i} \varphi_j^{(i)} d\sigma = |\partial D_j| \delta_{ij}. \quad (3.6)$$

This implies that φ_1 and φ_2 are linearly independent in \mathcal{H} . Since \mathcal{H}_0 has codimension 2 in \mathcal{H} and $\frac{1}{2}\mathbb{I} - \mathbb{K}^*$ is invertible in \mathcal{H}_0 as mentioned before, the multiplicity of $1/2$ is 2. \square

Using exactly the same arguments one can generalize Theorem 3.2 to the case when there are N simply connected inclusions.

Theorem 3.3 *If there are N simply connected mutually disjoint inclusions, then the eigenvalue $1/2$ of \mathbb{K}^* has multiplicity N .*

4 Characterization of the gradient blow-up

Let φ_j , $j = 1, 2$, be the eigenfunctions of \mathbb{K}^* corresponding to $1/2$ introduced in the proof of Theorem 3.2. Because of (3.6), if we define \mathbf{g} by

$$\mathbf{g} := \frac{1}{|\partial D_1|} \varphi_1 - \frac{1}{|\partial D_2|} \varphi_2, \quad (4.1)$$

then we have

$$\int_{\partial D_1} g^{(1)} d\sigma = 1, \quad \int_{\partial D_2} g^{(2)} d\sigma = -1. \quad (4.2)$$

Define

$$q(\mathbf{x}) := \mathcal{S}_{D_1}[g^{(1)}](\mathbf{x}) + \mathcal{S}_{D_2}[g^{(2)}](\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d \setminus (D_1 \cup D_2). \quad (4.3)$$

Then, q is the solution to

$$\begin{cases} \Delta q = 0 & \text{in } \mathbb{R}^d \setminus \overline{(D_1 \cup D_2)}, \\ q = \text{constant} & \text{on } \partial D_i, \ i = 1, 2, \\ \int_{\partial D_1} \frac{\partial q}{\partial \nu^{(1)}} \Big|_+ d\sigma = 1, \quad \int_{\partial D_2} \frac{\partial q}{\partial \nu^{(2)}} \Big|_+ d\sigma = -1, \\ q(\mathbf{x}) = O(|\mathbf{x}|^{1-d}) & \text{as } |\mathbf{x}| \rightarrow \infty, \end{cases} \quad (4.4)$$

and

$$\begin{bmatrix} \frac{\partial q}{\partial \nu^{(1)}} \Big|_+ \\ \frac{\partial q}{\partial \nu^{(2)}} \Big|_+ \end{bmatrix} = \mathbf{g}. \quad (4.5)$$

In fact, since $\mathbb{K}^*[\mathbf{g}] = \frac{1}{2}\mathbf{g}$, we have

$$\left[\frac{\partial}{\partial \nu^{(1)}} (\mathcal{S}_{D_1}[g^{(1)}](\mathbf{x}) + \mathcal{S}_{D_2}[g^{(2)}]) \Big|_- \right] = \left(-\frac{1}{2}\mathbb{I} + \mathbb{K}^* \right) \begin{bmatrix} g^{(1)} \\ g^{(2)} \end{bmatrix} = 0.$$

Thus we have the second line in (4.4) and (4.5). The third line in (4.4) follows from (4.5). Because of (4.2), we have

$$\begin{aligned} q(\mathbf{x}) &= \int_{\partial D_1} \Gamma(\mathbf{x} - \mathbf{y}) g^{(1)}(\mathbf{y}) \, d\sigma(\mathbf{y}) + \int_{\partial D_2} \Gamma(\mathbf{x} - \mathbf{y}) g^{(2)}(\mathbf{y}) \, d\sigma(\mathbf{y}) \\ &= \int_{\partial D_1} [\Gamma(\mathbf{x} - \mathbf{y}) - \Gamma(\mathbf{x} - \mathbf{y}_0)] g^{(1)}(\mathbf{y}) \, d\sigma(\mathbf{y}) \\ &\quad + \int_{\partial D_2} [\Gamma(\mathbf{x} - \mathbf{y}) - \Gamma(\mathbf{x} - \mathbf{y}_0)] g^{(2)}(\mathbf{y}) \, d\sigma(\mathbf{y}) \end{aligned}$$

for any fixed \mathbf{y}_0 . Since

$$|\Gamma(\mathbf{x} - \mathbf{y}) - \Gamma(\mathbf{x} - \mathbf{y}_0)| \leq C|\mathbf{x}|^{1-d} \quad \text{as } |\mathbf{x}| \rightarrow \infty,$$

we have the last line in (4.4).

It is known that if D_1 and D_2 are disks, then the singular function q is given by (1.9) and completely characterizes the blow-up of ∇u (see [13]). We have the following theorem as a generalization. Here and throughout this paper $u|_{\partial D_j}$ denotes the (constant) value of u on ∂D_j .

Proposition 4.1 *Assume that D_1 and D_2 are simply connected domains in \mathbb{R}^d , $d = 2, 3$, with $\mathcal{C}^{1,\alpha}$ boundaries for some $\alpha > 0$. The solution u to (1.5) can be written as*

$$u(\mathbf{x}) = c_\epsilon q(\mathbf{x}) + b(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d \setminus (D_1 \cup D_2), \quad (4.6)$$

where the constant c_ϵ is given by

$$c_\epsilon := \frac{u|_{\partial D_1} - u|_{\partial D_2}}{q|_{\partial D_1} - q|_{\partial D_2}} = \frac{\sum_{j=1}^2 \int_{\partial D_j} h \frac{\partial q}{\partial \nu^{(j)}} \Big|_+ \, d\sigma}{q|_{\partial D_1} - q|_{\partial D_2}}, \quad (4.7)$$

and

$$\|\nabla b\|_{L^\infty(\Omega)} \leq C \quad (4.8)$$

for some C independent of ϵ . Here $\Omega = \Omega_1 \setminus (D_1 \cup D_2)$ and Ω_1 is an open set containing $\overline{D_1 \cup D_2}$.

Remark 4.2 *The constant c_ϵ in (4.7) may depend on $\epsilon := \text{dist}(D_1, D_2)$, but is bounded independently of ϵ , and hence the singular function q determines the blow-up of ∇u , if D_1 and D_2 are strictly convex and have $\mathcal{C}^{2,\alpha}$ smooth boundaries. This fact will be proved in the next section.*

Proof of Proposition 4.1. Let u be the solution to (1.5) and let

$$\mathbf{u} := \begin{bmatrix} u|_{\partial D_1} \\ u|_{\partial D_2} \end{bmatrix}, \quad \mathbf{h} := \begin{bmatrix} h|_{\partial D_1} \\ h|_{\partial D_2} \end{bmatrix}.$$

Since u is constant on ∂D_1 and ∂D_2 , it follows from the third line of (4.4) that

$$u|_{\partial D_1} - u|_{\partial D_2} = \int_{\partial D_1} u \frac{\partial q}{\partial \nu^{(1)}}|_+ d\sigma + \int_{\partial D_2} u \frac{\partial q}{\partial \nu^{(2)}}|_+ d\sigma = \langle \mathbf{u}, \mathbf{g} \rangle.$$

The representation (2.4) implies that

$$\mathbf{u} = \mathbf{h} + \mathbb{S}[\varphi]$$

and $\varphi \in \mathcal{H}_0$. Therefore we have

$$\langle \mathbf{u}, \mathbf{g} \rangle = \langle \mathbf{h} + \mathbb{S}[\varphi], \mathbf{g} \rangle = \langle \mathbf{h}, \mathbf{g} \rangle + \langle \varphi, \mathbb{S}[\mathbf{g}] \rangle = \langle \mathbf{h}, \mathbf{g} \rangle.$$

So we have the second identity in (4.7).

Let

$$b(\mathbf{x}) = u(\mathbf{x}) - c_\epsilon q(\mathbf{x}) = u(\mathbf{x}) - \frac{u|_{\partial D_1} - u|_{\partial D_2}}{q|_{\partial D_1} - q|_{\partial D_2}} q(\mathbf{x}).$$

Then we have

$$b|_{\partial D_1} = b|_{\partial D_2}.$$

So, one can show following the same lines of the proof of Theorem 2.1 in [13] that (4.8) holds. This completes the proof. \square

It will be shown that the gradient of the singular function q defined as a single layer potential of an eigenfunction \mathbf{g} of \mathbb{K}^* blows up as $\epsilon = \text{dist}(D_1, D_2) \rightarrow 0$. We now show that another eigenfunction does not contribute to the blow-up. Let $\mathbf{f} = (f^{(1)}, f^{(2)})^T \in \mathcal{H}$ be an eigenfunction of \mathbb{K}^* orthogonal to \mathbf{g} with respect to the inner product (2.12), namely,

$$\langle \mathbb{S}[\mathbf{f}], \mathbf{g} \rangle = 0.$$

Then (4.2) shows that $\lambda^{(1)} = \lambda^{(2)}$ where $\lambda^{(j)}$ is the j -th component of the (constant) vector $\mathbb{S}[\mathbf{f}]$. It implies that the function v defined by

$$v(\mathbf{x}) := \mathcal{S}_{D_1}[f^{(1)}](\mathbf{x}) + \mathcal{S}_{D_2}[f^{(2)}](\mathbf{x})$$

is constant on ∂D_j , $j = 1, 2$, and satisfies

$$v|_{\partial D_1} = v|_{\partial D_2}.$$

So, $|\nabla v|$ stays bounded regardless of ϵ .

5 Estimates of the potential difference

We assume for the rest of this paper that D_1 and D_2 are strictly convex domains in \mathbb{R}^2 with $\mathcal{C}^{2,\alpha}$ boundaries for some $\alpha > 0$. Let \mathbf{z}_1 and \mathbf{z}_2 be points on ∂D_1 and ∂D_2 , respectively, such that

$$|\mathbf{z}_1 - \mathbf{z}_2| = \text{dist}(D_1, D_2) = \epsilon. \quad (5.1)$$

We prove the following proposition.

Proposition 5.1 *Let u be the solution to (1.5), c_ϵ be the constant defined by (4.7), and κ_j be the curvature of ∂D_j at \mathbf{z}_j for $j = 1, 2$. Then c_ϵ is bounded independently of ϵ and*

$$u|_{\partial D_1} - u|_{\partial D_2} = -\frac{c_\epsilon}{\sqrt{2\pi}} \sqrt{\kappa_1 + \kappa_2} \sqrt{\epsilon} (1 + O_\alpha), \quad (5.2)$$

where

$$O_\alpha = \begin{cases} O(\epsilon^{\alpha/2}) & \text{if } \alpha \in (0, 1), \\ O(|\sqrt{\epsilon} \ln \epsilon|) & \text{if } \alpha = 1. \end{cases} \quad (5.3)$$

We prove Proposition 5.1 after a sequence of lemmas.

Let B_j be the osculating disk to D_j at \mathbf{z}_j so that its radius r_j equals to $1/\kappa_j$. Let q_B be the singular function associated with B_1 and B_2 , i.e., the solution to (4.4) with D_1 and D_2 replaced with B_1 and B_2 . Recall that q_B is given explicitly by

$$q_B(\mathbf{x}) = \frac{1}{2\pi} (\ln |\mathbf{x} - \mathbf{p}_1| - \ln |\mathbf{x} - \mathbf{p}_2|), \quad (5.4)$$

where $\mathbf{p}_1 \in B_1$ and $\mathbf{p}_2 \in B_2$ are the unique fixed points of repeated reflections $R_1 R_2$ and $R_2 R_1$, respectively. We emphasize that q_B is harmonic in $\mathbb{R}^2 \setminus \{\mathbf{p}_1, \mathbf{p}_2\}$.

For the rest of this paper we assume that $\mathbf{z}_1 = (-\epsilon/2, 0)$ and $\mathbf{z}_2 = (\epsilon/2, 0)$ after translation and rotation if necessary, so that the centers of B_1 and B_2 are on the real axis. In this case, \mathbf{p}_1 and \mathbf{p}_2 are of the forms $\mathbf{p}_j = (p_j, 0)$, $j = 1, 2$, and it is proved in [23] that

$$p_j = (-1)^j \sqrt{2} \sqrt{\frac{r_1 r_2}{r_1 + r_2}} \sqrt{\epsilon} + O(\epsilon), \quad j = 1, 2. \quad (5.5)$$

It is also proved using (5.5) that

$$q_B|_{\partial B_1} - q_B|_{\partial B_2} = -\left(\frac{1}{\sqrt{2\pi}} \sqrt{\frac{r_1 + r_2}{r_1 r_2}}\right) \sqrt{\epsilon} + O(\epsilon) \text{ as } \epsilon \rightarrow 0. \quad (5.6)$$

Using (5.5) one can see that (1.16) holds.

Lemma 5.2 *There is a constant C independent of ϵ such that*

$$\left| \frac{\partial q}{\partial \nu^{(j)}} \Big|_+ (\mathbf{x}) \right| \leq C |\nabla q_B(\mathbf{x})| \quad \text{for all } \mathbf{x} \in \partial D_j, \quad j = 1, 2. \quad (5.7)$$

Proof. We only prove (5.7) for $j = 2$ since the case for $j = 1$ can be treated in the exactly same way. We first assume that D_2 is a disk so that $B_2 = D_2$. Let B'_1 be a disk of radius r'_1 and let B''_1 be a disk of radius r''_1 (r'_1 and r''_1 are independent of ϵ) such that $B'_1 \subset D_1 \subset B''_1$ and $\mathbf{z}_1 \in \partial B'_1 \cap \partial B''_1$. Let q' and q'' be the solution to (4.4) with D_1 replaced with B'_1 and B''_1 , respectively. Then the monotonic property [21, Lemma 2.4] yields

$$0 \leq -\frac{\partial q}{\partial \nu^{(2)}} \Big|_+ (\mathbf{x}) \leq -\left(\frac{q''|_{\partial B''_1} - q''|_{\partial B_2}}{q'|_{\partial B'_1} - q'|_{\partial B_2}}\right) \frac{\partial q'}{\partial \nu^{(2)}} \Big|_+ (\mathbf{x}), \quad \mathbf{x} \in \partial D_2. \quad (5.8)$$

Because of (5.6), there is a constant C independent of ϵ such that

$$\frac{q''|_{\partial B''_1} - q''|_{\partial B_2}}{q'|_{\partial B'_1} - q'|_{\partial B_2}} \leq C, \quad (5.9)$$

so we have

$$\left| \frac{\partial q}{\partial \nu^{(2)}} \Big|_+ (\mathbf{x}) \right| \leq C |\nabla q'(\mathbf{x})|, \quad \mathbf{x} \in \partial D_2. \quad (5.10)$$

Note that

$$q'(\mathbf{x}) = \frac{1}{2\pi} (\ln |\mathbf{x} - \mathbf{p}'_1| - \ln |\mathbf{x} - \mathbf{p}'_2|), \quad (5.11)$$

where \mathbf{p}'_1 and \mathbf{p}'_2 are the fixed points of the repeated reflections with respect to $\partial B'_1$ and ∂B_2 . Using (5.5) one can show that

$$|\nabla q'(\mathbf{x})| \leq C |\nabla q_B(\mathbf{x})|, \quad \mathbf{x} \in \partial D_2 \quad (5.12)$$

for some constant C independent of ϵ . So we have (5.7) provided that D_2 is a disk.

If D_2 is not a disk, we may use a conformal mapping to make it a disk. In fact, if $\tilde{\Psi}$ is a conformal mapping from $\mathbb{R}^2 \cup \{\infty\} \setminus \overline{U}$ onto $\mathbb{R}^2 \cup \{\infty\} \setminus \overline{\tilde{D}_2}$ where U is the unit disk, then $\tilde{\Psi}$ can be extended up to ∂U as a C^1 function. Therefore, there are constants C_1 and C_2 such that

$$C_1 \leq |\nabla \tilde{\Psi}(\mathbf{x})| \leq C_2 \quad \text{for all } \mathbf{x} \in \partial U. \quad (5.13)$$

Let

$$\Psi = \tilde{\Psi} + \mathbf{a}_2, \quad (5.14)$$

where \mathbf{a}_2 is defined in (1.4). Then Ψ is a conformal mapping from $\mathbb{R}^2 \setminus \overline{U}$ onto $\mathbb{R}^2 \setminus \overline{\tilde{D}_2}$ and satisfies (5.13). Moreover, by [23, Appendix] and a combination with a linear fractional transformation, we can also assume that there are two disks B'_1 and B''_1 of radii independent of ϵ such that $B'_1 \subset \Psi^{-1}(D_1) \subset B''_1$ and $\partial B'_1 \cap \partial B''_1$ contains the point on $\partial \Psi^{-1}(D_1)$ which is the closest to U . Thus we can apply the same argument as above to $q \circ \Psi - c_q$ to obtain (5.7), where $c_q = \lim_{\mathbf{x} \rightarrow \infty} q \circ \Psi(\mathbf{x})$. This completes the proof. \square

Lemma 5.3 *There exists a positive δ_0 (independent of ϵ) such that if $\mathbf{x} \in \partial D_j$ and $|\mathbf{x} - \mathbf{z}_j| \leq \delta_0$, then*

$$|q_B(\mathbf{x}) - q_{B|\partial B_j}| \leq C\sqrt{\epsilon}|\mathbf{x} - \mathbf{z}_j|^\alpha, \quad (5.15)$$

and

$$\left| \frac{\partial q}{\partial \nu^{(j)}} \Big|_+ (\mathbf{x}) \right| \leq C \frac{\sqrt{\epsilon}}{|\mathbf{x} - \mathbf{z}_j|^2 + \epsilon}. \quad (5.16)$$

For any point $\mathbf{x} \in \partial D_j$ with $|\mathbf{x} - \mathbf{z}_j| > \delta_0$,

$$|q_B(\mathbf{x}) - q_{B|\partial B_j}| \leq C\sqrt{\epsilon} \quad (5.17)$$

and

$$\left| \frac{\partial q}{\partial \nu^{(j)}} \Big|_+ (\mathbf{x}) \right| \leq C\sqrt{\epsilon}. \quad (5.18)$$

Here, $j = 1, 2$, and the constants C are independent of ϵ and δ_0 .

Proof. Assume that $j = 2$ without loss of generality. There exists $\delta_0 > 0$ (independent of ϵ) and functions $x_2, x_B : [-\delta_0, \delta_0] \rightarrow \mathbb{R}$ such that $x_2(0) = \epsilon/2$, $x'_2(0) = 0$, $x_B(0) = \epsilon/2$, $x'_B(0) = 0$, and ∂D_2 and ∂B_2 are graphs of x_2 and x_B for $|y| \leq \delta_0$, i.e., $(x_2(y), y) \in \partial D_2$ and $(x_B(y), y) \in \partial B_2$. We then have

$$|x_2(y)| \leq C|y|^2 \quad \text{and} \quad |x_2(y) - x_B(y)| \leq C|y|^{2+\alpha} \quad (5.19)$$

for some constant C since D_2 and B_2 are osculating at \mathbf{z}_2 . Since the fixed points \mathbf{p}_1 and \mathbf{p}_2 of the repeated reflections are on the x -axis, we may write $\mathbf{p}_j = (p_j, 0)$, $j = 1, 2$, and (5.5) holds.

If $|y| \leq \delta_0$, then

$$\begin{aligned} & \left| q_B(x_2(y), y) - q_B|_{\partial B_2} \right| \\ &= \frac{1}{2\pi} \left(\ln |(x_2(y) - p_1, y)| - \ln |(x_2(y) - p_2, y)| \right) \\ & \quad - \frac{1}{2\pi} \left(\ln |(x_B(y) - p_1, y)| - \ln |(x_B(y) - p_2, y)| \right). \end{aligned}$$

If $0 < |y| < \epsilon^{\frac{1}{2(2+\alpha)}} \leq \delta_0$, there exists x_* between $x_2(y)$ and $x_B(y)$ such that

$$\begin{aligned} & \left| q_B(x_2(y), y) - q_B|_{\partial B_2} \right| \\ & \leq C|x_2(y) - x_B(y)| \left| \frac{(x_* - p_1)}{(x_* - p_1)^2 + y^2} - \frac{(x_* - p_2)}{(x_* - p_2)^2 + y^2} \right| \\ & \leq C|x_2(y) - x_B(y)| |p_1 - p_2| \frac{1}{y^2} \\ & \leq C|y|^{2+\alpha} (|y|^{2+\alpha} + \sqrt{\epsilon}) \frac{1}{y^2} \leq C\sqrt{\epsilon}|y|^\alpha, \end{aligned}$$

where the second to last inequality follows from (5.5).

If $\epsilon^{\frac{1}{2(2+\alpha)}} \leq |y| \leq \delta_0$, there exists p_* between p_1 and p_2 such that

$$\begin{aligned} & \left| q_B(x_2(y), y) - q_B|_{\partial B_2} \right| \\ & \leq C|p_1 - p_2| \left| \frac{(x_2(y) - p_*)}{(x_2(y) - p_*)^2 + y^2} - \frac{(x_B(y) - p_*)}{(x_B(y) - p_*)^2 + y^2} \right| \\ & \leq C|p_1 - p_2| |x_2(y) - x_B(y)| \frac{1}{y^2} \\ & \leq C\sqrt{\epsilon} (|y|^{2+\alpha} + \sqrt{\epsilon}) \frac{1}{y^2} \leq C\sqrt{\epsilon}|y|^\alpha, \end{aligned}$$

where the the second to last inequality holds because of (5.19).

If $|(x, y) - (\epsilon/2, 0)| > \delta_0$, one can easily see from (5.5) that

$$|q_B(x, y)| \leq C|p_1 - p_2| \leq C\sqrt{\epsilon},$$

and hence we have

$$\left| q_B(x_2(y), y) - q_B|_{\partial B_2} \right| \leq |q_B(x_2(y), y)| + \left| q_B|_{\partial B_2} \right| \leq C\sqrt{\epsilon}.$$

Now we estimate $\frac{\partial q}{\partial \nu^{(2)}}|_+$ on ∂D_2 . By (5.7), we have

$$\left| \frac{\partial q}{\partial \nu^{(2)}}|_+ \right| \leq C|\nabla q_B| \text{ on } \partial D_2.$$

Suppose that $|y| \leq \delta_0$. Then, we have

$$\begin{aligned} \left| \frac{\partial q}{\partial \nu^{(2)}} \Big|_+(x_2(y), y) \right| &\leq C \left| \frac{\partial q_B}{\partial x}(x_2(y), y) \right| \\ &\leq C \left| \frac{(x_2(y) - p_1)}{(x_2(y) - p_1)^2 + y^2} - \frac{(x_2(y) - p_2)}{(x_2(y) - p_2)^2 + y^2} \right|. \end{aligned}$$

If $|y| < \sqrt{\epsilon}$, then $|x_2(y) - p_j| > C\sqrt{\epsilon}$ for $j = 1, 2$, and thus we obtain

$$|\nabla q(x_2(y), y)| \leq C \frac{1}{\sqrt{\epsilon}}.$$

If $\sqrt{\epsilon} \leq |y| \leq \delta_0$, then it follows that

$$|\nabla q(x_2(y), y)| \leq C |p_1 - p_2| \frac{1}{y^2} \leq C \frac{\sqrt{\epsilon}}{y^2}.$$

For (x, y) with $|(x, y) - (\epsilon/2, 0)| > \delta_0$, we have

$$|\nabla q_B(x, y)| \leq C\sqrt{\epsilon},$$

and (5.18) follows. This completes the proof. \square

Lemma 5.4 *We have*

$$q|_{\partial D_1} - q|_{\partial D_2} = -\frac{1}{\sqrt{2\pi}} \sqrt{\kappa_1 + \kappa_2} \sqrt{\epsilon} + \begin{cases} O(\epsilon^{(\alpha+1)/2}) & \text{if } \alpha \in (0, 1), \\ O(|\epsilon \ln \epsilon|) & \text{if } \alpha = 1, \end{cases} \quad (5.20)$$

as $\epsilon \rightarrow 0$.

Proof. We prove that

$$q|_{\partial D_1} - q|_{\partial D_2} = q_B|_{\partial B_1} - q_B|_{\partial B_2} + \begin{cases} O(\epsilon^{(\alpha+1)/2}) & \text{if } \alpha \in (0, 1), \\ O(|\epsilon \ln \epsilon|) & \text{if } \alpha = 1, \end{cases} \quad (5.21)$$

as $\epsilon \rightarrow 0$. Then (5.20) follows from (5.6).

Let

$$v(\mathbf{x}) := q(\mathbf{x}) - q_B(\mathbf{x}). \quad (5.22)$$

Since

$$\int_{\partial D_i} \frac{\partial q_B}{\partial \nu^{(i)}} \Big|_+ d\sigma = \int_{\partial B_i} \frac{\partial q_B}{\partial \nu^{(i)}} \Big|_+ d\sigma, \quad i = 1, 2,$$

the function v satisfies

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}, \\ v(\mathbf{x}) + q_B(\mathbf{x}) - q_B|_{\partial B_i} = \text{constant on } \partial D_i, \\ \int_{\partial D_i} \frac{\partial v}{\partial \nu^{(i)}} \Big|_+ d\sigma = 0, \quad i = 1, 2, \\ v(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \rightarrow \infty. \end{cases} \quad (5.23)$$

Then, we have

$$q|_{\partial D_1} - q|_{\partial D_2} - (q_B|_{\partial B_1} - q_B|_{\partial B_2}) = - \sum_{i=1}^2 (-1)^i (v(\mathbf{x}) + q_B(\mathbf{x}) - q_B|_{\partial B_i}) \Big|_{\partial D_i}.$$

We then obtain from the third line in (4.4) and the second line in (5.23) that

$$q|_{\partial D_1} - q|_{\partial D_2} - (q_B|_{\partial B_1} - q_B|_{\partial B_2}) = \sum_{i=1}^2 \int_{\partial D_i} (v + q_B - q_B|_{\partial B_i}) \frac{\partial q}{\partial \nu^{(i)}} \Big|_+ d\sigma.$$

An integration by parts and the third line in (5.23) yield

$$q|_{\partial D_1} - q|_{\partial D_2} - (q_B|_{\partial B_1} - q_B|_{\partial B_2}) = \sum_{i=1}^2 \int_{\partial D_i} (q_B - q_B|_{\partial B_i}) \frac{\partial q}{\partial \nu^{(i)}} \Big|_+ d\sigma.$$

Let

$$\begin{aligned} & \left| \int_{\partial D_i} (q_B - q_B|_{\partial B_i}) \frac{\partial q}{\partial \nu^{(i)}} \Big|_+ d\sigma \right| \\ &= \left| \int_{|\mathbf{x}-\mathbf{z}_i| \leq \delta_0} + \int_{|\mathbf{x}-\mathbf{z}_i| > \delta_0} (q_B - q_B|_{\partial B_i}) \frac{\partial q}{\partial \nu^{(i)}} \Big|_+ d\sigma \right| := I_1 + I_2. \end{aligned}$$

Using (5.17) and (5.18) we have

$$|I_2| \leq C\epsilon.$$

To estimate I_1 , let N be the smallest integer such that $\delta_0 \leq 2^N \sqrt{\epsilon}$. We then have from (5.15) and (5.16) that

$$\begin{aligned} |I_1| &\leq \left| \int_{|\mathbf{x}-\mathbf{z}_i| \leq \sqrt{\epsilon}} + \sum_{j=1}^N \int_{2^{j-1}\sqrt{\epsilon} < |\mathbf{x}-\mathbf{z}_i| \leq 2^j \sqrt{\epsilon}} (q_B - q_B|_{\partial B_i}) \frac{\partial q}{\partial \nu^{(i)}} \Big|_+ d\sigma \right| \\ &\leq C \left[\int_{|\mathbf{x}-\mathbf{z}_i| \leq \sqrt{\epsilon}} |\mathbf{x} - \mathbf{z}_i| d\sigma + \epsilon \sum_{j=1}^N \int_{2^{j-1}\sqrt{\epsilon} < |\mathbf{x}-\mathbf{z}_i| \leq 2^j \sqrt{\epsilon}} \frac{1}{|\mathbf{x} - \mathbf{z}_i|^{2-\alpha}} d\sigma \right] \\ &\leq \begin{cases} C_\alpha(\epsilon + \epsilon^{\frac{1+\alpha}{2}}) \leq 2C_\alpha \epsilon^{\frac{1+\alpha}{2}} & \text{if } \alpha \in (0, 1), \\ C(\epsilon + \epsilon N) \leq C\epsilon \ln \frac{1}{\epsilon} & \text{if } \alpha = 1. \end{cases} \end{aligned}$$

This completes the proof. \square

Proof of Proposition 5.1. We first prove that c_ϵ is bounded independently of ϵ . For that we prove that

$$|\langle \mathbf{h}, \mathbf{g} \rangle| = \left| \sum_{j=1}^2 \int_{\partial D_j} h \frac{\partial q}{\partial \nu^{(j)}} \Big|_+ d\sigma \right| \leq C\sqrt{\epsilon}. \quad (5.24)$$

We still assume that $\mathbf{z}_1 = (-\epsilon/2, 0)$ and $\mathbf{z}_2 = (\epsilon/2, 0)$. Pick a point $(c, 0) \in D_2$ where c is independent of ϵ , and let

$$\psi(x, y) := \frac{c^2 y}{(x - c)^2 + y^2}. \quad (5.25)$$

Then, ψ is harmonic except at $(c, 0)$ and $\psi(x, y) = O(|(x, y)|^{-1})$ as $|(x, y)| \rightarrow \infty$. Since q is constant on ∂D_j , $j = 1, 2$, we have by the divergence theorem that

$$\sum_{j=1}^2 \int_{\partial D_j} \psi \frac{\partial q}{\partial \nu^{(j)}}|_+ d\sigma = \sum_{j=1}^2 \int_{\partial D_j} \frac{\partial \psi}{\partial \nu^{(j)}}|_+ q d\sigma = \sum_{j=1}^2 q|_{\partial D_j} \int_{\partial D_j} \frac{\partial \psi}{\partial \nu^{(j)}}|_+ d\sigma = 0. \quad (5.26)$$

Moreover, one can easily see that there is a constant $C > 0$ such that

$$|\psi(x, y) - y| \leq C(x^2 + y^2) \quad \text{for all } (x, y) \in \mathbb{R}^2 \setminus (D_1 \cup D_2). \quad (5.27)$$

Therefore, we have from Taylor's theorem that

$$\left| h(x, y) - h(0, 0) - \frac{\partial h}{\partial y}(0, 0)\psi(x, y) \right| \leq C(|x| + y^2) \quad (5.28)$$

for all $(x, y) \in \partial D_1 \cup \partial D_2$.

Because of the third line in (4.4) and (5.26), we have

$$\sum_{j=1}^2 \int_{\partial D_j} h \frac{\partial q}{\partial \nu^{(j)}}|_+ d\sigma = \sum_{j=1}^2 \int_{\partial D_j} \left(h - h(0, 0) - \frac{\partial h}{\partial y}(0, 0)\psi \right) \frac{\partial q}{\partial \nu^{(j)}}|_+ d\sigma.$$

Let

$$\int_{\partial D_1} \left(h - h(0, 0) - \frac{\partial h}{\partial y}(0, 0)\psi \right) \frac{\partial q}{\partial \nu^{(1)}}|_+ d\sigma = \int_{|\mathbf{x}-\mathbf{z}_1| \leq \delta_0} + \int_{|\mathbf{x}-\mathbf{z}_1| > \delta_0} := I_1 + I_2.$$

It follows from (5.16) and (5.28) that

$$|I_1| \leq C \int_{|y| \leq \delta_0} \frac{\sqrt{\epsilon}(|x_1(y)| + y^2)}{y^2 + \epsilon} d\sigma \leq C' \int_{|y| \leq \delta_0} \frac{\sqrt{\epsilon}y^2}{y^2 + \epsilon} d\sigma \leq C''\sqrt{\epsilon}.$$

By (5.18), we have

$$|I_2| \leq C\sqrt{\epsilon}.$$

Therefore, we have

$$\left| \int_{\partial D_1} \left(h - h(0, 0) - \frac{\partial h}{\partial y}(0, 0)\psi \right) \frac{\partial q}{\partial \nu^{(1)}}|_+ d\sigma \right| \leq C\sqrt{\epsilon}.$$

Similarly, we can show that

$$\left| \int_{\partial D_2} \left(h - h(0, 0) - \frac{\partial h}{\partial y}(0, 0)\psi \right) \frac{\partial q}{\partial \nu^{(2)}}|_+ d\sigma \right| \leq C\sqrt{\epsilon}.$$

Hence we obtain (5.24). We now infer from (5.6) and Lemma 5.4 that c_ϵ is bounded regardless of ϵ .

Since

$$u|_{\partial D_1} - u|_{\partial D_2} = c_\epsilon(q|_{\partial D_1} - q|_{\partial D_2})$$

by (4.7), (5.2) follows from Lemma 5.4. \square

6 Estimates of the gradient- Proof of Theorem 1.1

Proposition 5.1 and (5.24) show that

$$c_\epsilon = -\frac{\sqrt{2\pi}\langle \mathbf{h}, \mathbf{g} \rangle}{\sqrt{\epsilon(\kappa_1 + \kappa_2)}}(1 + O_\alpha), \quad (6.1)$$

where

$$O_\alpha = \begin{cases} O(\epsilon^{\alpha/2}) & \text{if } \alpha \in (0, 1), \\ O(|\sqrt{\epsilon} \ln \epsilon|) & \text{if } \alpha = 1. \end{cases} \quad (6.2)$$

So, Theorem 1.1 is an immediate consequence of Proposition 4.1 and the following proposition.

Proposition 6.1 *We have*

$$q(\mathbf{x}) = a_\epsilon q_B(\mathbf{x}) + v(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2 \setminus (D_1 \cup D_2), \quad (6.3)$$

where

$$a_\epsilon := \frac{q|_{\partial D_1} - q|_{\partial D_2}}{q_B|_{\partial B_1} - q_B|_{\partial B_2}} = 1 + \begin{cases} O(\epsilon^{\alpha/2}) & \text{if } \alpha \in (0, 1), \\ O(|\sqrt{\epsilon} \ln \epsilon|) & \text{if } \alpha = 1, \end{cases} \quad (6.4)$$

and

$$\|\nabla v\|_{L^\infty(\mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)})} \leq C \quad (6.5)$$

for some constant C independent of ϵ .

We first fix notation. We suppose that $\mathbf{z}_1 = (-\epsilon/2, 0)$ and $\mathbf{z}_2 = (\epsilon/2, 0)$ as before. There exists $\delta_0 > 0$ (independent of ϵ) and functions $x_1, x_2 : [-\delta_0, \delta_0] \rightarrow \mathbb{R}$ such that $x_1(0) = -\epsilon/2$, $x_1'(0) = 0$, $x_2(0) = \epsilon/2$, $x_2'(0) = 0$, and ∂D_1 and ∂D_2 are graphs of x_1 and x_2 for $|y| \leq \delta_0$, i.e., $(x_1(y), y) \in \partial D_1$ and $(x_2(y), y) \in \partial D_2$. Since D_1 and D_2 are strictly convex, x_1 is strictly concave and x_2 is strictly convex. For $\delta \leq \delta_0$, let

$$\Pi_\delta := \{(x, y) \in \mathbb{R}^2 \setminus (D_1 \cup D_2) \mid x_1(y) < x < x_2(y), |y| \leq \delta\}.$$

To prove Proposition 6.1, we need the following result whose proof will be given in the last section.

Lemma 6.2 *If v is a bounded harmonic function in $\mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}$ satisfying*

$$\left| \frac{\partial^2 v}{\partial \tau^2}(x_i(y), y) \right| \leq M|y|^{\alpha-1} \quad \text{for } |y| \leq \delta_0, \quad (6.6)$$

$$\left\| \frac{\partial^2 v}{\partial \tau^2} \right\|_{L^\infty((\partial D_1 \cup \partial D_2) \setminus \partial \Pi_{\delta_0})} \leq M \quad (6.7)$$

for some constant M (independent of ϵ), and

$$v(\mathbf{z}_1) = \frac{\partial}{\partial \tau} v(\mathbf{z}_1) = v(\mathbf{z}_2) = \frac{\partial}{\partial \tau} v(\mathbf{z}_2) = 0, \quad (6.8)$$

where $\frac{\partial}{\partial \tau}$ is the tangential derivative on ∂D_i , then there exists a constant C independent of $\epsilon > 0$ such that

$$\|\nabla v\|_{L^\infty(\mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)})} \leq C. \quad (6.9)$$

We also need the following lemma.

Lemma 6.3 *There exists a positive constant M independent of ϵ such that*

$$\left| \frac{\partial^2 q_B}{\partial \tau^2}(x_i(y), y) \right| \leq M|y|^{\alpha-1} \quad \text{for } |y| \leq \delta_0, \quad (6.10)$$

and

$$\left\| \frac{\partial^2 q_B}{\partial \tau^2} \right\|_{L^\infty(\partial D_i \setminus \partial \Pi_{\delta_0})} \leq M \quad (6.11)$$

for $i = 1, 2$.

Proof. We prove (6.10) and (6.11) for $i = 2$. We use the same notation as in the proof of Lemma 5.3: ∂B_2 are given by $(x_B(y), y)$ for $|y| \leq \delta_0$. Let $\mathbf{x}_2(y) = (x_2(y), y)$ and $\mathbf{x}_B(y) = (x_B(y), y)$.

Note that

$$\frac{\partial^2 q_B}{\partial \tau^2} = \frac{\partial^2}{\partial \tau^2} (q_B - q_B(\mathbf{z}_2)) \approx \frac{d^2}{dy^2} (q_B(\mathbf{x}_2(y)) - q_B(\mathbf{x}_B(y)))$$

if $|y| \leq \delta_0$. Straightforward computations yield

$$\begin{aligned} & \frac{d}{dy} (q_B(\mathbf{x}_2(y)) - q_B(\mathbf{x}_B(y))) \\ &= \frac{1}{2\pi} \sum_{i=1}^2 (-1)^{i+1} \left(\frac{(\mathbf{x}_2(y) - \mathbf{p}_i) \cdot \mathbf{x}'_2(y)}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2} - \frac{(\mathbf{x}_B(y) - \mathbf{p}_i) \cdot \mathbf{x}'_B(y)}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2} \right), \end{aligned}$$

and

$$\begin{aligned} & \frac{d^2}{dy^2} (q_B(\mathbf{x}_2(y)) - q_B(\mathbf{x}_B(y))) \\ &= \frac{1}{2\pi} \sum_{i=1}^2 (-1)^{i+1} \left(\frac{|\mathbf{x}'_2(y)|^2}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2} - \frac{|\mathbf{x}'_B(y)|^2}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2} \right) \\ & \quad + \frac{1}{2\pi} \sum_{i=1}^2 (-1)^{i+1} \left(\frac{(\mathbf{x}_2(y) - \mathbf{p}_i) \cdot \mathbf{x}''_2(y)}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2} - \frac{(\mathbf{x}_B(y) - \mathbf{p}_i) \cdot \mathbf{x}''_B(y)}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2} \right) \\ & \quad + \frac{1}{\pi} \sum_{i=1}^2 (-1)^i \left(\frac{((\mathbf{x}_2(y) - \mathbf{p}_i) \cdot \mathbf{x}'_2(y))^2}{|\mathbf{x}_2(y) - \mathbf{p}_i|^4} - \frac{((\mathbf{x}_B(y) - \mathbf{p}_i) \cdot \mathbf{x}'_B(y))^2}{|\mathbf{x}_B(y) - \mathbf{p}_i|^4} \right) \\ & := I_1 + I_2 + I_3. \end{aligned}$$

To estimate I_1 , I_2 and I_3 , we make some preliminary computations. Since B_2 and D_2 are osculating at \mathbf{z}_2 , we have

$$|x_2(y) - x_B(y)| \leq C|y|^{2+\alpha}. \quad (6.12)$$

Since $x_2(0) = x_B(0) = \epsilon/2$ and $x'_2(0) = x'_B(0) = 0$, we have

$$|x_2(y)| + |x_B(y)| \leq C(y^2 + \epsilon), \quad |x'_2(y)| + |x'_B(y)| \leq C|y|, \quad |x''_2(y)| + |x''_B(y)| \leq C, \quad (6.13)$$

and

$$|x_2(y) - p_i| \leq C|y|^2 + \sqrt{\epsilon}, \quad |x_B(y) - p_i| \leq C|y|^2 + \sqrt{\epsilon}, \quad j = 1, 2. \quad (6.14)$$

It is worth mentioning that the constant C may differ at each appearance. We also have

$$|x'_2(y) - x'_B(y)| \leq C|y|^{1+\alpha}, \quad (6.15)$$

and

$$|\mathbf{x}_2(y) - \mathbf{p}_i|^2 \geq C(y^2 + \epsilon), \quad |\mathbf{x}_B(y) - \mathbf{p}_i|^2 \geq C(y^2 + \epsilon), \quad j = 1, 2. \quad (6.16)$$

To estimate I_1 , we write

$$\begin{aligned} & \left| \frac{|\mathbf{x}'_2(y)|^2}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2} - \frac{|\mathbf{x}'_B(y)|^2}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2} \right| \\ & \leq |\mathbf{x}'_2(y)|^2 \left| \frac{1}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2} - \frac{1}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2} \right| + \frac{||\mathbf{x}'_2(y)|^2 - |\mathbf{x}'_B(y)|^2|}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2}. \end{aligned}$$

Using (6.12), (6.14) and (6.16) we get

$$\begin{aligned} \left| \frac{1}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2} - \frac{1}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2} \right| & \leq \frac{|x_2(y) - x_B(y)| |x_2(y) + x_B(y) - 2p_i|}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2 |\mathbf{x}_B(y) - \mathbf{p}_i|^2} \\ & \leq C \frac{\sqrt{\epsilon}|y|^{2+\alpha} + |y|^{4+\alpha}}{y^4 + \epsilon^2} \leq C \frac{1}{|y|^{1-\alpha}}. \end{aligned} \quad (6.17)$$

We then use (6.15) to arrive at

$$\frac{||\mathbf{x}'_2(y)|^2 - |\mathbf{x}'_B(y)|^2|}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2} \leq \frac{|x'_2(y)^2 - x'_B(y)^2|}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2} \leq C \frac{|y|^{1+\alpha}}{y^2 + \epsilon} \leq C \frac{1}{|y|^{1-\alpha}}.$$

Thus we have

$$|I_1| \leq C|y|^{\alpha-1}.$$

It follows from (6.17) that

$$\begin{aligned} & \left| \frac{(\mathbf{x}_2(y) - \mathbf{p}_i) \cdot \mathbf{x}''_2(y)}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2} - \frac{(\mathbf{x}_B(y) - \mathbf{p}_i) \cdot \mathbf{x}''_B(y)}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2} \right| \\ & \leq \left| \frac{\mathbf{x}_2(y) \cdot \mathbf{x}''_2(y)}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2} \right| + \left| \frac{\mathbf{x}_B(y) \cdot \mathbf{x}''_B(y)}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2} \right| \\ & \quad + \left| \frac{\mathbf{p}_i \cdot (\mathbf{x}''_2(y) - \mathbf{x}''_B(y))}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2} \right| + \left| (\mathbf{p}_i \cdot \mathbf{x}''_B(y)) \left(\frac{1}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2} - \frac{1}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2} \right) \right| \\ & \leq C \left(\frac{y^2 + \epsilon}{y^2 + \epsilon} + \frac{y^2 + \epsilon}{y^2 + \epsilon} + \frac{\sqrt{\epsilon}|y|^\alpha}{y^2 + \epsilon} + \sqrt{\epsilon} \frac{\sqrt{\epsilon}|y|^{2+\alpha} + |y|^{4+\alpha}}{y^4 + \epsilon^2} \right) \leq C|y|^{\alpha-1}, \end{aligned}$$

and hence

$$|I_2| \leq C|y|^{\alpha-1}.$$

To estimate I_3 , we first write

$$\begin{aligned} & \left| \frac{((\mathbf{x}_2(y) - \mathbf{p}_i) \cdot \mathbf{x}'_2(y))^2}{|\mathbf{x}_2(y) - \mathbf{p}_i|^4} - \frac{((\mathbf{x}_B(y) - \mathbf{p}_i) \cdot \mathbf{x}'_B(y))^2}{|\mathbf{x}_B(y) - \mathbf{p}_i|^4} \right| \\ & \leq \left| \frac{(\mathbf{x}_2(y) - \mathbf{p}_i) \cdot \mathbf{x}'_2(y)}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2} + \frac{(\mathbf{x}_B(y) - \mathbf{p}_i) \cdot \mathbf{x}'_B(y)}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2} \right| \\ & \quad \times \left| \frac{(\mathbf{x}_2(y) - \mathbf{p}_i) \cdot \mathbf{x}'_2(y)}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2} - \frac{(\mathbf{x}_B(y) - \mathbf{p}_i) \cdot \mathbf{x}'_B(y)}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2} \right|. \end{aligned}$$

One can easily see from (6.16) that

$$\left| \frac{(\mathbf{x}_2(y) - \mathbf{p}_i) \cdot \mathbf{x}'_2(y)}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2} + \frac{(\mathbf{x}_B(y) - \mathbf{p}_i) \cdot \mathbf{x}'_B(y)}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2} \right| \leq \frac{C|y|}{y^2 + \epsilon}. \quad (6.18)$$

Note that

$$\begin{aligned} & \frac{(\mathbf{x}_2(y) - \mathbf{p}_i) \cdot \mathbf{x}'_2(y)}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2} - \frac{(\mathbf{x}_B(y) - \mathbf{p}_i) \cdot \mathbf{x}'_B(y)}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2} \\ &= \frac{(x_2(y) - p_i)x'_2(y)}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2} - \frac{(x_B(y) - p_i)x'_B(y)}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2} + \frac{y}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2} - \frac{y}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2} \\ &= \frac{x_2(y)x'_2(y)}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2} - \frac{x_B(y)x'_B(y)}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2} - \frac{p_i(x'_2(y) - x'_B(y))}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2} \\ & \quad + (p_i x'_B(y) - y) \left(\frac{1}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2} - \frac{1}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2} \right). \end{aligned}$$

We estimate each term using (6.12)-(6.16) to have

$$\begin{aligned} & \left| \frac{(\mathbf{x}_2(y) - \mathbf{p}_i) \cdot \mathbf{x}'_2(y)}{|\mathbf{x}_2(y) - \mathbf{p}_i|^2} - \frac{(\mathbf{x}_B(y) - \mathbf{p}_i) \cdot \mathbf{x}'_B(y)}{|\mathbf{x}_B(y) - \mathbf{p}_i|^2} \right| \\ & \leq C \left[\frac{|y|(|y|^2 + \epsilon)}{y^2 + \epsilon} + \frac{|y|(|y|^2 + \epsilon)}{y^2 + \epsilon} + \frac{\sqrt{\epsilon}|y|^{1+\alpha}}{y^2 + \epsilon} + (\sqrt{\epsilon} + 1)|y| \frac{\sqrt{\epsilon}|y|^{2+\alpha} + |y|^{2+\alpha}(y^2 + \epsilon)}{y^4 + \epsilon^2} \right] \\ & \leq C|y|^\alpha. \end{aligned}$$

Combining this estimates with (6.18) we obtain

$$|I_3| \leq C|y|^{\alpha-1}.$$

If $\mathbf{x} \in \partial D_2$ satisfies $|\mathbf{x} - (\epsilon/2, 0)| > \delta_0$, it can be easily seen that

$$\left| \frac{\partial^2 v}{\partial \tau^2}(\mathbf{x}) \right| \leq M.$$

The proof is complete. \square

Proof of Proposition 6.1. Note that the second identity in (6.4) follows from Lemma 5.4, and it shows in particular that a_ϵ is bounded regardless of ϵ .

Let

$$v(\mathbf{x}) = q(\mathbf{x}) - a_\epsilon q_B(\mathbf{x}), \quad (6.19)$$

and

$$w(\mathbf{x}) := \frac{1}{a_\epsilon}(v(\mathbf{x}) - v(\mathbf{z}_2)). \quad (6.20)$$

Then one can see from the definition (6.4) of a_ϵ and (6.19) that

$$w(\mathbf{x}) = q_B(\mathbf{z}_i) - q_B(\mathbf{x}), \quad \mathbf{x} \in \partial D_i, \quad i = 1, 2. \quad (6.21)$$

Since D_i and B_i are osculating at \mathbf{z}_i , we have in particular

$$w(\mathbf{z}_1) = \frac{\partial w}{\partial \tau}(\mathbf{z}_1) = w(\mathbf{z}_2) = \frac{\partial w}{\partial \tau}(\mathbf{z}_2) = 0.$$

It follows from Lemma 6.3 and Lemma 6.2 that

$$\|\nabla w\|_{L^\infty(\mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)})} \leq C.$$

Since a_ϵ is bounded, we obtain (6.5). This completes the proof. \square

7 The insulating case

In this section we deal with the case when the inclusions are insulating, namely, the problem (1.20). We closely follow the argument provided in [13].

Let h^\perp be a harmonic conjugate of h , i.e., $h + ih^\perp$ is analytic. Let u^\perp be the solution to (1.5) with h^\perp in place of h . Then the solution u to (1.20) is a harmonic conjugate of u^\perp in $\mathbb{R}^2 \setminus \overline{D_1 \cup D_2}$. By Theorem 1.1, we have

$$u^\perp(\mathbf{x}) = -\frac{\sqrt{2\pi}\langle \mathbf{h}^\perp, \mathbf{g} \rangle}{\sqrt{\epsilon(\kappa_1 + \kappa_2)}} \beta_\epsilon q_B(\mathbf{x}) + r(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2 \setminus (D_1 \cup D_2), \quad (7.1)$$

where β_ϵ is a constant of the form

$$\beta_\epsilon = \begin{cases} 1 + O(\epsilon^{\alpha/2}) & \text{if } \alpha \in (0, 1), \\ 1 + O(|\sqrt{\epsilon} \ln \epsilon|) & \text{if } \alpha = 1, \end{cases} \quad \text{as } \epsilon \rightarrow 0. \quad (7.2)$$

Let $\arg : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow [-\pi, \pi)$ be the argument function with a branch cut along the negative real axis, where $\mathbf{x} = (x_1, x_2)$ is identified with $x_1 + ix_2$. Define

$$q_B^\perp(\mathbf{x}) = \frac{1}{2\pi} \left(\arg(\mathbf{x} - \mathbf{p}_1) - \arg(\mathbf{x} - \mathbf{p}_2) - \arg(\mathbf{x} - \mathbf{c}_1) + \arg(\mathbf{x} - \mathbf{c}_2) \right), \quad (7.3)$$

where \mathbf{c}_j is the center of B_j , $j = 1, 2$. Note that q_B^\perp is a harmonic function well defined in $\mathbb{R}^2 \setminus \overline{(B_1 \cup B_2)}$ since the jump discontinuity of the argument function across the branch cut is cancelled out owing to the fact $\mathbf{p}_j, \mathbf{c}_j \in B_j$, $j = 1, 2$. Since $\arg(\mathbf{x} - \mathbf{p}_1) - \arg(\mathbf{x} - \mathbf{p}_2)$ is a harmonic conjugate of q_B except on the branch cut and $|\nabla(\arg(\mathbf{x} - \mathbf{c}_1) - \arg(\mathbf{x} - \mathbf{c}_2))|$ is bounded independently of ϵ , we arrive at the following result.

Theorem 7.1 *Let u be the solution to (1.20). Under the same hypothesis as in Theorem 1.1, we have*

$$u(\mathbf{x}) = -\frac{\sqrt{2\pi}\langle \mathbf{h}^\perp, \mathbf{g} \rangle}{\sqrt{\epsilon(\kappa_1 + \kappa_2)}} \beta_\epsilon q_B^\perp(\mathbf{x}) + r(\mathbf{x}), \quad \mathbf{x} \in \Omega \setminus (D_1 \cup D_2), \quad (7.4)$$

where β_ϵ is a constant of the form (7.2) and

$$\|\nabla r\|_{L^\infty(\Omega)} \leq C \quad (7.5)$$

for some C independent of ϵ .

8 Boundary value problems

Let Ω be a bounded domain in \mathbb{R}^2 with \mathcal{C}^2 boundary. Suppose that Ω contains two perfectly conducting inclusions D_j , $j = 1, 2$, which have $\mathcal{C}^{2,\alpha}$ boundaries for some $\alpha > 0$. We assume that the inclusions are away from $\partial\Omega$, namely, there is a constant c_0 such that

$$\text{dist}(D_j, \partial\Omega) \geq c_0, \quad j = 1, 2. \quad (8.1)$$

In this section we consider the following boundary value problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus \overline{D_1 \cup D_2}, \\ u = \text{constant on } \partial D_j, \quad j = 1, 2, \\ \int_{\partial D_j} \frac{\partial u}{\partial \nu(j)} ds = 0, \quad j = 1, 2, \end{cases} \quad (8.2)$$

with either Dirichlet or Neumann boundary conditions on $\partial\Omega$.

Define a harmonic function h in Ω by (1.22). This h plays the role of h in the free space problem (1.5) and we obtain the following result using exactly the same arguments as those in Theorem 1.1.

Theorem 8.1 *Let u be the solution to (8.2) and let h be the function defined by (1.22). Then, under the same hypothesis as in Theorem 1.1, we have*

$$u(\mathbf{x}) = -\frac{\sqrt{2}\pi\langle \mathbf{h}, \mathbf{g} \rangle}{\sqrt{\epsilon(\kappa_1 + \kappa_2)}} \alpha_\epsilon q_B(\mathbf{x}) + r(\mathbf{x}), \quad \mathbf{x} \in \Omega \setminus (D_1 \cup D_2), \quad (8.3)$$

where α_ϵ is a constant of the form

$$\alpha_\epsilon = \begin{cases} 1 + O(\epsilon^{\alpha/2}) & \text{if } \alpha \in (0, 1), \\ 1 + O(|\sqrt{\epsilon} \ln \epsilon|) & \text{if } \alpha = 1, \end{cases} \quad \text{as } \epsilon \rightarrow 0, \quad (8.4)$$

and

$$\|\nabla r\|_{L^\infty(\Omega)} \leq C \quad (8.5)$$

for some C independent of ϵ .

One can also obtain a similar result for the insulating boundary value problem.

9 Proof of Lemma 6.2

We use the same notation as in Section 6.

Let $\chi_2(\mathbf{x})$ be a smooth function on ∂D_2 such that

$$\begin{cases} \chi_2(x_2(y), y) = 1 & \text{for } |y| \leq \frac{\delta_0}{2}, \\ 0 \leq \chi_2(x_2(y), y) \leq 1 & \text{for } \frac{\delta_0}{2} \leq |y| \leq \delta_0, \\ \chi_2(\mathbf{x}) = 0 & \text{otherwise.} \end{cases} \quad (9.1)$$

As before we denote the tangential derivative on ∂D_j by $\frac{\partial}{\partial \tau}$. Let $g_{2+}(\mathbf{x})$ and $g_{2-}(\mathbf{x})$ be non-negative functions defined for $\mathbf{x} = (x_2(y), y) \in \partial D_2$, $|y| \leq \delta_0$, such that

$$g_{2+}(\mathbf{z}_2) = \frac{\partial g_{2+}}{\partial \tau}(\mathbf{z}_2) = g_{2-}(\mathbf{z}_2) = \frac{\partial g_{2-}}{\partial \tau}(\mathbf{z}_2) = 0 \quad (9.2)$$

and

$$\frac{\partial^2 g_{2+}}{\partial \tau^2}(\mathbf{x}) = \max \left\{ \frac{\partial^2 v(\mathbf{x})}{\partial \tau^2}, 0 \right\}, \quad \frac{\partial^2 g_{2-}}{\partial \tau^2}(\mathbf{x}) = \max \left\{ -\frac{\partial^2 v(\mathbf{x})}{\partial \tau^2}, 0 \right\}. \quad (9.3)$$

Then, g_{2+} and g_{2-} satisfy

$$g_{2+}(x_2(y), y) - g_{2-}(x_2(y), y) = v(x_2(y), y), \quad |y| \leq \delta_0. \quad (9.4)$$

Let V_{2+} , V_{2-} , \tilde{V}_{2+} and \tilde{V}_{2-} be bounded harmonic functions in $\mathbb{R}^2 \setminus \overline{D_2}$ which satisfy the following Dirichlet boundary conditions on ∂D_2 :

$$\begin{cases} V_{2+} = \chi_2 g_{2+} \\ V_{2-} = \chi_2 g_{2-} \\ \tilde{V}_{2+} = (1 - \chi_2) \max\{v, 0\} \\ \tilde{V}_{2-} = -(1 - \chi_2) \min\{v, 0\} \end{cases} \quad \text{on } \partial D_2. \quad (9.5)$$

Then by the maximum principle, V_{2+} , V_{2-} , \tilde{V}_{2+} and \tilde{V}_{2-} are non-negative and satisfy

$$V_{2+} - V_{2-} + \tilde{V}_{2+} - \tilde{V}_{2-} = v \quad \text{on } \partial D_2. \quad (9.6)$$

Let v_{2+} , v_{2-} , \tilde{v}_{2+} and \tilde{v}_{2-} be bounded harmonic functions in $\mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}$ which satisfy the following Dirichlet conditions on ∂D_1 and ∂D_2 :

$$v_{2+} = v_{2-} = \tilde{v}_{2+} = \tilde{v}_{2-} = 0 \quad \text{on } \partial D_1, \quad (9.7)$$

and

$$\begin{cases} v_{2+} = V_{2+} \\ v_{2-} = V_{2-} \\ \tilde{v}_{2+} = \tilde{V}_{2+} \\ \tilde{v}_{2-} = \tilde{V}_{2-} \end{cases} \quad \text{on } \partial D_2. \quad (9.8)$$

Since $0 \leq v_{2\pm} \leq V_{2\pm}$ and $0 \leq \tilde{v}_{2\pm} \leq \tilde{V}_{2\pm}$ on ∂D_1 and ∂D_2 , we have

$$0 \leq v_{2\pm} \leq V_{2\pm} \quad \text{and} \quad 0 \leq \tilde{v}_{2\pm} \leq \tilde{V}_{2\pm} \quad \text{in } \mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}. \quad (9.9)$$

Let

$$w_2 := v_{2+} - v_{2-} + \tilde{v}_{2+} - \tilde{v}_{2-}. \quad (9.10)$$

Then w_2 is a bounded harmonic function in $\mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}$ and is such that

$$w_2 = 0 \quad \text{on } \partial D_1, \quad w_2 = v \quad \text{on } \partial D_2. \quad (9.11)$$

In the same way, non-negative bounded harmonic functions v_{1+} , v_{1-} , \tilde{v}_{1+} and \tilde{v}_{1-} in $\mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}$ can be defined so that $w_1 := v_{1+} - v_{1-} + \tilde{v}_{1+} - \tilde{v}_{1-}$ satisfies

$$w_1 = v \quad \text{on } \partial D_1, \quad w_1 = 0 \quad \text{on } \partial D_2. \quad (9.12)$$

Then, we have from the uniqueness of the Dirichlet problem that

$$v = w_1 + w_2 \quad \text{in } \mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}. \quad (9.13)$$

We first estimate $|\nabla v_{2+}|$. Thanks to (6.6) and (6.7), we have

$$\left| \frac{\partial^2 V_{2+}}{\partial \tau^2}(x_2(y), y) \right| \leq C|y|^{\alpha-1} \quad \text{for } |y| \leq \delta_0, \quad (9.14)$$

and

$$\left\| \frac{\partial^2 V_{2+}}{\partial \tau^2} \right\|_{L^\infty(\partial D_2 \setminus \partial \Pi_{\delta_0})} \leq C. \quad (9.15)$$

Since $V_{2+}(\mathbf{z}_2) = \frac{\partial}{\partial \tau} V_{2+}(\mathbf{z}_2) = 0$, we have $\|V_{2+}\|_{C^{1,\alpha}(\partial D_2)} \leq C$. A standard regularity estimate for harmonic functions yields, in particular,

$$\|V_{2+}\|_{C^1(\mathbb{R}^2 \setminus D_2)} \leq C. \quad (9.16)$$

If $(x, y) \in \Pi_{\delta_0}$, then

$$|(x, y) - (x(y), y)| \leq C(y^2 + \epsilon).$$

Thus we obtain from (9.16) and the mean value theorem that

$$0 \leq V_{2+}(x, y) \leq C(y^2 + \epsilon), \quad (x, y) \in \Pi_{\delta_0}. \quad (9.17)$$

It then follows from (9.9) that

$$0 \leq v_{2+}(x, y) \leq C(y^2 + \epsilon), \quad (x, y) \in \Pi_{\delta_0}. \quad (9.18)$$

Let $d(\mathbf{x}) := \text{dist}(\mathbf{x}, \partial D_2)$ and $B_r(\mathbf{x})$ be the disk of radius r centered at \mathbf{x} . Since $v_{2+} = 0$ on ∂D_1 and ∂D_1 is $C^{2,\alpha}$, by a standard elliptic regularity estimate we have

$$|\nabla v_{2+}(\mathbf{x})| \leq \frac{C\|v_{2+}\|_{L^\infty(B_{d(\mathbf{x})}(\mathbf{x}) \cap (\mathbb{R}^2 \setminus (D_1 \cup D_2)))}}{d(\mathbf{x})}. \quad (9.19)$$

If $\mathbf{x} = (x, y) \in \Pi_{\delta_0/2}$ and $x < 0$, then there are c_1 and c_2 such that

$$c_1(y^2 + \epsilon) \leq d(\mathbf{x}) \leq c_2(y^2 + \epsilon).$$

If $(x', y') \in B_{d(\mathbf{x})}(\mathbf{x}) \cap (\mathbb{R}^2 \setminus (D_1 \cup D_2))$, then $|y'| \leq |y| + c_2(y^2 + \epsilon)$, thus we have

$$|v_{2+}(x', y')| \leq C(y'^2 + \epsilon) \leq C'(y^2 + \epsilon).$$

We then get from (9.19) that

$$|\nabla v_{2+}(\mathbf{x})| \leq C.$$

If $\mathbf{x} = (x, y) \in \Pi_{\delta_0/2}$ and $x \geq 0$, then we can apply the same argument using the fact that $V_{2+} - v_{2+} = 0$ on ∂D_2 to obtain

$$|\nabla(V_{2+} - v_{2+})(\mathbf{x})| \leq C.$$

We then obtain using (9.16) that

$$|\nabla v_{2+}(\mathbf{x})| \leq C.$$

So, we have

$$|\nabla v_{2+}(\mathbf{x})| \leq C, \quad \mathbf{x} \in \Pi_{\delta_0/2}. \quad (9.20)$$

We now estimate $|\nabla \tilde{v}_{2+}|$. Since $\|v\|_{C^{1,\alpha}(\partial D_2)} \leq C$, we first obtain from the maximum principle that

$$\|\tilde{V}_{2+}\|_{L^\infty(\mathbb{R}^2 \setminus D_2)} \leq C \quad (9.21)$$

for some C independent of ϵ . Since

$$\tilde{V}_{2+}(x_2(y), y) = 0 \quad \text{if } |y| \leq \delta_0/2, \quad (9.22)$$

we have

$$|\nabla \tilde{V}_{2+}(\mathbf{x})| \leq C \quad (9.23)$$

for all $\mathbf{x} \in \mathbb{R}^2 \setminus D_2$ satisfying

$$\text{dist}(\mathbf{x}, \partial D_2 \setminus \partial \Pi_{\delta_0/2}) \geq \delta_0/4. \quad (9.24)$$

In particular, we have

$$\sup_{\mathbf{x} \in \Pi_{\delta_0/4}} |\nabla \tilde{V}_{2+}(\mathbf{x})| \leq C. \quad (9.25)$$

It follows from (9.22) and (9.25) that

$$0 \leq \tilde{V}_{2+}(x, y) \leq C(y^2 + \epsilon), \quad (x, y) \in \Pi_{\delta_0/4},$$

and from (9.9) that

$$0 \leq \tilde{v}_{2+}(x, y) \leq C(y^2 + \epsilon), \quad (x, y) \in \Pi_{\delta_0/4}.$$

Since $\tilde{v}_{2+}(x_1(y), y) = \tilde{v}_{2+}(x_2(y), y) = 0$ if $|y| \leq \delta_0/2$, we may apply the same argument as for v_{2+} to obtain

$$|\nabla \tilde{v}_{2+}(\mathbf{x})| \leq C, \quad \mathbf{x} \in \Pi_{\delta_0/8}. \quad (9.26)$$

In exactly the same way, one can show that

$$|\nabla v_{2-}(\mathbf{x})| + |\nabla \tilde{v}_{2-}(\mathbf{x})| \leq C, \quad \mathbf{x} \in \Pi_{\delta_0/8}. \quad (9.27)$$

Therefore, we have

$$\sup_{\mathbf{x} \in \Pi_{\delta_0/8}} |\nabla w_2(\mathbf{x})| \leq C. \quad (9.28)$$

If $\mathbf{x} \in \partial D_1 \setminus \partial \Pi_{\delta_0/8}$, then $d(\mathbf{x}) \geq C$ for some C independent of ϵ . Since $w_2 = 0$ on ∂D_1 and w_2 is bounded, we obtain

$$\sup_{\mathbf{x} \in \partial D_1 \setminus \partial \Pi_{\delta_0/8}} |\nabla w_2(\mathbf{x})| \leq C. \quad (9.29)$$

Let

$$V = V_{2+} - V_{2-} + \tilde{V}_{2+} - \tilde{V}_{2-} \quad \text{in } \mathbb{R}^2 \setminus D_2. \quad (9.30)$$

Since $V - w_2 = 0$ on ∂D_2 , it follows that

$$|\nabla(V - w_2)(\mathbf{x})| \leq C$$

for $\mathbf{x} \in \partial D_2 \setminus \partial \Pi_{\delta_0/8}$. Since $\|V\|_{C^{1,\alpha}(\partial D_2)}$ is bounded, $\|\nabla V\|_{L^\infty(\mathbb{R}^2 \setminus D_2)}$ is bounded, so we have

$$\sup_{\mathbf{x} \in \partial D_2 \setminus \partial \Pi_{\delta_0/8}} |\nabla w_2(\mathbf{x})| \leq C. \quad (9.31)$$

Inequalities (9.28), (9.29) and (9.31) imply that

$$\sup_{\mathbf{x} \in \partial((\mathbb{R}^2 \setminus (D_1 \cup D_2)) \setminus \Pi_{\delta_0/8})} |\nabla w_2(\mathbf{x})| \leq C.$$

We then obtain from the maximum principle that

$$\sup_{\mathbf{x} \in (\mathbb{R}^2 \setminus (D_1 \cup D_2)) \setminus \Pi_{\delta_0/8}} |\nabla w_2(\mathbf{x})| \leq C.$$

Combining this with (9.28), we readily get

$$\sup_{\mathbf{x} \in \mathbb{R}^2 \setminus (D_1 \cup D_2)} |\nabla w_2(\mathbf{x})| \leq C. \quad (9.32)$$

One can show in exactly the same way (by switching the roles of D_1 and D_2) that

$$\sup_{\mathbf{x} \in \mathbb{R}^2 \setminus (D_1 \cup D_2)} |\nabla w_1(\mathbf{x})| \leq C. \quad (9.33)$$

Thus we have (6.9) and the proof is complete. \square

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